

# The Aharonov Casher Effect: The Case of $g \neq 2$

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## Abstract

The Aharonov Casher effect predicts the existence in two dimensions of  $\left[\frac{\Phi}{2\pi}\right] - 1$  bounded zero modes associated with a magnetic flux  $\Phi$ . Aharonov and Casher discussed the case of gyromagnetic factor equals 2, we will discuss the general case of any gyromagnetic factor. As a simple model, we study the case where the magnetic field lies in a thin annulus. First we examine the wavefunctions of the zero-energy bounded states, predicted by the Aharonov Casher Effect for electrons with gyromagnetic ratio equal 2. We then calculate the wave function and energies for a gyromagnetic ratio  $g \neq 2$ . We give the dependence of the bound states energies on  $g$  and the angular momentum. Finally, we provide an order of magnitude estimations for the binding energies.

## 1 Aharonov Casher zero modes

In this section we will find explicitly the Aharonov Casher zero modes for a magnetic field which lives inside a thin annulus. First we write the Pauli equation for an electron in a plane. We take the magnetic field to be in the direction normal to the plane and the electron's spin to be along the field. Taking units where  $\hbar = c = 2m_e = 1$  we get:

$$\left((-i\nabla - A(r)e)^2 + \frac{g}{2}B(r)e\right)\psi = 0 \quad (1)$$

From now on, we will measure magnetic flux in units of  $\frac{1}{e}$ . In our case, the entire magnetic flux lives on a infinitesimal thin circle of radius  $R$ :

$$B(r) = \delta(R - r) \cdot \frac{\Phi}{2\pi R}$$

We will work in the Coulomb gauge:

$$\begin{aligned} r < R & : \quad A = 0 \\ r > R & : \quad A(r) = (\Phi/2\pi r)\hat{\theta} \end{aligned}$$

and in angular coordinates:

$$r < R : \quad \left(-\frac{1}{r}\partial_r r \partial_r + (-i\partial_\theta/r)^2\right)\psi_{in} = 0 \quad (2)$$

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$$r > R : \quad \left( -\frac{1}{r} \partial_r r \partial_r + (-i \partial_\theta / r - \Phi / 2\pi r)^2 \right) \psi_{out} = 0 \quad (3)$$

The Pauli equation (1) around at  $r = R$  gives the jump conditions between  $\psi_{in}$  and  $\psi_{out}$ . In a small enough neighborhood of  $r = R$  we can keep only second derivatives in  $r$  and the Stern-Gerlach term, while neglecting all the other terms with respect to them, to get:  $(-\partial_r^2 - \frac{g}{2} B) \psi = 0$ . Integrating  $\partial_r^2$  in a small neighborhood around  $r = R$ , we get the jump condition of  $\psi$  between the two domains,  $r < R$  and  $r > R$ :

$$(\partial_r \psi_{out} - \partial_r \psi_{in})|_{r=R} = -\frac{g}{2} \frac{\Phi}{2\pi R} \psi \quad (4)$$

In order to find the wavefunction in the two domains, we will use the separation of variables:  $\psi = f(r) e^{i\theta m}$  ( $m \in \mathbb{Z}$ ), and get:<sup>1</sup>.

$$\psi_{in} = C_0 r^m e^{im\theta} + D_0 r^{-m} e^{im\theta}$$

$$\psi_{out} = E_0 r^{m - \frac{\Phi}{2\pi}} e^{im\theta} + F_0 r^{-(m - \frac{\Phi}{2\pi})} e^{im\theta}$$

From the restrictions of continuity, normalization and matching of the jump condition we must take  $D_0 = F_0 = 0$  and  $m \geq 0$ . The jump condition gives us  $C_0 = E_0$ , which also give us continuity. Note that the existence of the Aharonov and Casher zero mode seems to be very sensitive to the jump condition, determined by the value of  $g$ . The remaining wavefunction is normalizable if, and only if,  $\psi_{out}$  is normalizable. Therefore we require:

$$\int_R^\infty |\psi|^2 2\pi r dr = \int_R^\infty \left( r^{(m - \frac{\Phi}{2\pi})} \right)^2 2\pi r dr < \infty$$

or  $m < \frac{\Phi}{2\pi} - 1$ . This gives us the  $\left\lceil \frac{\Phi}{2\pi} \right\rceil - 1$  different solutions predicted by Aharonov and Casher.

## 2 Bounded states with correction to the gyromagnetic factor

In this section we will find the wavefunctions and binding energies for the case of  $g > 2$  (the case of  $g < 2$  does not yield a normalizable wavefunctions). Using the same separation of variables as before:  $\psi = f(r) e^{i\theta m}$ , we get:

$$r < R : \quad \left( -\frac{1}{r} \partial_r r \partial_r + m^2 / r^2 \right) f(r)_{in} = E f(r)_{in}$$

$$r > R : \quad \left( -\frac{1}{r} \partial_r r \partial_r + (m - (\Phi/2\pi))^2 / r^2 \right) f(r)_{out} = E f(r)_{out}$$

Looking at  $E < 0$ , we express  $f(r)$  as a linear combination of the modified Bessel functions of the first and second kind ( $I_n, K_n$ ):

$$\begin{aligned} f(r)_{in} &= C_1 I_m(\sqrt{-E}r) + C_2 K_m(\sqrt{-E}r) \\ f(r)_{out} &= C_3 I_{(m - (\Phi/2\pi))}(\sqrt{-E}r) + C_4 K_{(m - (\Phi/2\pi))}(\sqrt{-E}r) \end{aligned}$$

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<sup>1</sup>For  $m = 0$ , the second term of  $\psi_{in}$  is  $D_0 \ln(r)$ , and is still not continuous at 0. A similar correction should be noted for the solutions of  $\psi_{out}$ , where the solution containing a logarithmic function is rejected for not being normalizable.

For normalization and continuity at 0, we must choose  $C_2 = C_3 = 0$ . From continuity at  $r = R$  we get:

$$\frac{C_1}{C_4} = \frac{K_{(m-(\Phi/2\pi))}(\sqrt{-E}R)}{I_m(\sqrt{-E}R)}$$

The binding energy  $E$  depends on the following parameters:  $n, \Phi, g$ . Naturally, we are interested in the function  $E(g)$ , for a value of  $g$  slightly above 2. It is technically simpler to examine  $g(m, \Phi, E)$ , which we get explicitly from jump condition (4):

$$g(E) = -(\partial_r \psi_{out} - \partial_r \psi_{in})|_{r=R} \frac{4\pi R}{\Phi} / \psi(R)$$

or:

$$g(E) = \left( \frac{\partial_r I_m(\sqrt{-E}r)|_{r=R}}{I_m(\sqrt{-E}R)} - \frac{\partial_r K_{(m-(\Phi/2\pi))}(\sqrt{-E}r)|_{r=R}}{K_{(m-(\Phi/2\pi))}(\sqrt{-E}R)} \right) \frac{4\pi R}{\Phi}$$

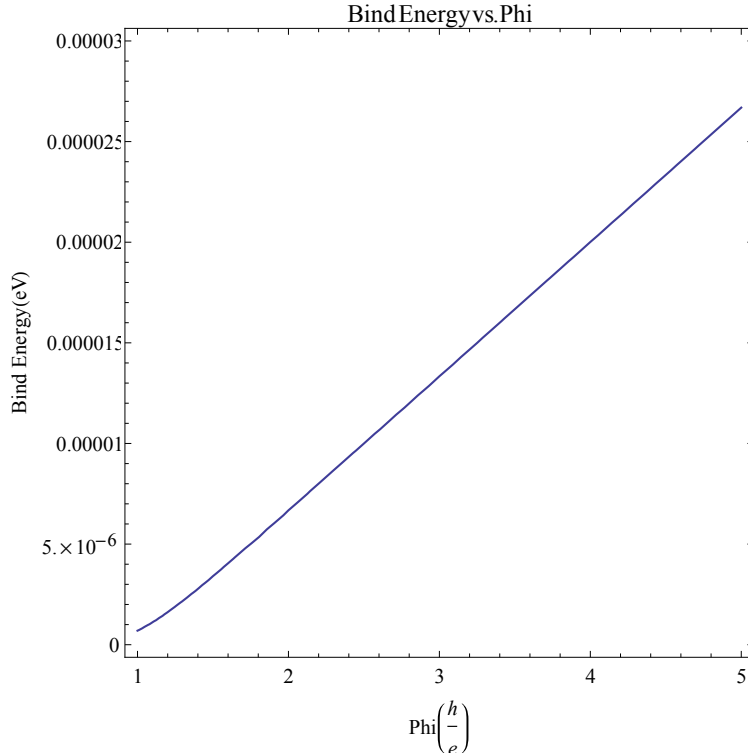
For  $m = 0$ , Expanding to first order in  $E$  we get:

$$g(E) = 2 - \frac{R^2 E}{2(\frac{\Phi}{2\pi} - 1)} + O(E^{\frac{\Phi}{2\pi}})$$

or:

$$E(g) = -\frac{(g-2) \cdot (\frac{\Phi}{2\pi} - 1)}{R^2} + O\left((g-2)^{\frac{\Phi}{2\pi}}\right)$$

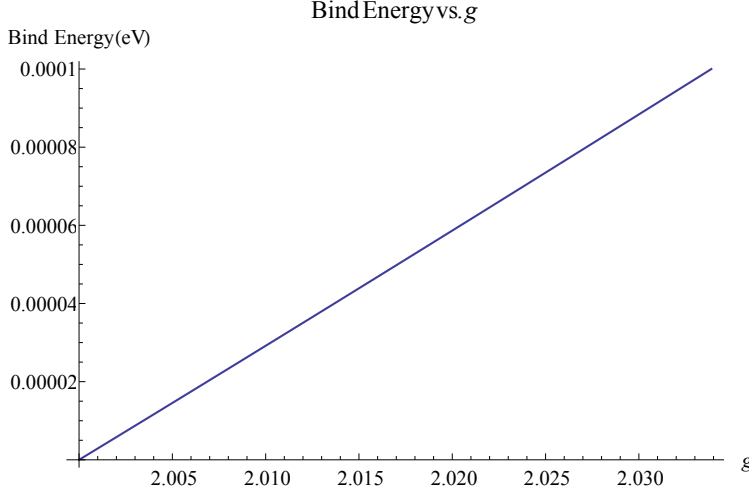
As an example, we can plot the electron's energy as function of the flux (noted as  $\phi$  in the graph), for the gyromagnetic ratio of an isolated electron given by QED and no orbital angular momentum. We plot it for  $R = 1/\alpha^2$  (or  $R = \frac{\hbar}{2m_e c \alpha^2} = 3.626 \cdot 10^{-9} \text{meter}$ )<sup>2</sup>.  $\Phi$  is shown in units of  $\frac{\Phi}{2\pi}$  (or  $\phi_{unit} = \frac{\hbar}{e} = 4.135 \cdot 10^{-15} \text{Wb}$ ) and  $E$  is of order of magnitude of  $m_e c^2 \alpha^5$  and shown in  $eV$ .



<sup>2</sup>See section 3.1 for the reason we chose this radius

We can see that the binding energy near  $\frac{\Phi}{2\pi} = 1$  is indeed of an order of magnitude less than the first order approximation, and that the slope is consistent with our approximation as well.

Another interesting relation is the binding energy as a function of the gyromagnetic ratio. For example, we plot it for a fixed  $\frac{\Phi}{2\pi} = 2$ :



We should also note two other interesting properties of those states: First, the critical flux for the emergence of a new bound state is somewhat lower with respect to the value given for  $g = 2$ . Second, the decay rate of the states far from the magnetic flux has a typical radius, in contrast with the  $g = 2$  power law case.

### 3 Order of magnitude

In this section we will work in m.k.s units.

#### 3.1 Estimation of the source of a unit magnetic flux:

A flux that is equal  $2\pi$  in the units we used previously, is equal  $\Phi_{unit} = 2\pi\hbar/e$  in the m.k.s. units. One possible way to produce such a flux is to use a collection of magnetic dipoles originating from an electron's orbital angular momentum or spin. We will use the semi-classical Bohr model in order to estimate how many such atoms will be needed in order to get a one unit of quantum flux. Our model is a current loop, with a current that matches a single electron with the velocity of  $\alpha c$ , where  $\alpha$  is the fine structure constant. This gives a current of  $I = \frac{e\alpha c}{2\pi a_0}$  in a loop of radius  $a_b$  (the Bohr radius). By the Biot Savart law we get:

$$B_{max} = \frac{e\alpha c}{2\pi a_b} \frac{\mu_0}{r}$$

For small radii around the current loop, the contribution to the flux from one side of the loop cancels the contribution from the other side. In a distance similar to the Bohr radius from the loop, there is no longer a similarity between the magnetic field in the two sides of the current loop so  $B_{center} \sim \frac{e\alpha c}{2\pi a_b} \frac{\mu_0}{a_b}$ . This typical value lasts for an area of about  $\pi a_b^2$  which gives us:

$$\Phi \sim B_{center} A = \frac{e\alpha}{\epsilon_0 c}$$

or:

$$\Phi/\Phi_q \sim \alpha^2$$

This means, we will need a magnetic tip with a cross section of order of magnitude of  $1/\alpha^2$  dipoles in order to produce one unit of quantum flux at the end of the tip.

A second option to produce such a flux, is to pass a strong magnetic flux through a superconductor of the second kind, such that the penetrations of the magnetic flux will be dense enough for spotting the effect.

### 3.2 Estimation of the binding energy

As we saw in section 2, the bind energy is of the order of magnitude of:

$$E \sim 2m_e c^2 (g - 2) \cdot \left( \frac{\Phi}{\Phi_q} - 1 \right) \frac{\left( \frac{\hbar}{2m_e c} \right)^2}{R^2}$$

For an electron with the QED vacuum correction to the gyromagnetic factor and a magnetic field in a thin annulus of radius  $r$ , we get:

$$E \sim \alpha \frac{\hbar^2}{m_e R^2} \left( \frac{\Phi}{\Phi_q} - 1 \right)$$

In comparison to the hydrogen atom bind energy  $E_h \sim \frac{\hbar^2}{a_b^2 m_e}$ , we get:

$$E/E_h \sim \alpha \frac{a_b^2}{R^2} \left( \frac{\Phi}{\Phi_q} - 1 \right)$$

Substituting a flux of a size of a few unit fluxes, according to the estimation from section (3.1), we get:

$$E/E_h \sim \alpha^3$$

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